On the Graph-Density of Random 0/1-Polytopes

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Abstract. Let $X_{d,n}$ be an n-element subset of $\{0,1\}^d$ chosen uniformly at random, and denote by $P_{d,n} := \operatorname{conv} X_{d,n}$ its convex hull. Let $\Delta_{d,n}$ be the density of the graph of $P_{d,n}$ (i.e., the number of one-dimensional faces of $P_{d,n}$ divided by $\binom{n}{2}$). Our main result is that, for any function $n(d)$, the expected value of $\Delta_{d,n(d)}$ converges (with $d \to \infty$) to one if, for some arbitrary $\varepsilon > 0$, $n(d) \le (\sqrt{2} - \varepsilon)^d$ holds for all large d, while it converges to zero if $n(d) \ge (\sqrt{2} + \varepsilon)^d$ holds for all large d.

1 Introduction

Polytopes whose vertices have coordinates in $\{0,1\}$ (*0/1-polytopes*) are the objects of study in large parts of polyhedral combinatorics (see [10]). Since that theory has started to grow, people have been interested in the *graphs* (defined by the vertices and the one-dimensional faces) of the polytopes under investigation. The main reason for this interest was, of course, the role played by polytope graphs with respect to linear programming and, in particular, the simplex algorithm.

Later it was recognized that the graphs of the 0/1-polytopes associated with certain combinatorial objects (such as matchings in a graph or bases of a matroid) might also yield good candidates for neighborhood structures with respect to the construction of random walks for random generation of the respective objects. A quite important (yet unsolved) problem arising in this context is the question whether the graphs of 0/1-polytopes have good expansion properties (see [3,5,7]).

We are short of knowledge on the graphs of *general* 0/1-polytopes [13]. Among the few exceptions are results about their diameters [8] and their cycle structures [9]. Particularly striking is the fact that several *special* 0/1-polytopes associated with combinatorial problems have quite dense graphs. The most prominent example for this is probably the *cut polytope* CUT_k, i.e., the convex hull of the characteristic vectors of those subsets of edges of the complete graph K_k that form cuts in K_k. Barahona and Mahjoub [1] proved that the graph of CUT_k is complete, i.e., its density equals one (where the *density* of a graph $G = (V,E)$ is $|E|/\binom{|V|}{2}$). Since the dimension of CUT_k is $d = \binom{k}{2}$ and

there are $n = 2^{k-1}$ cuts in K_k, the cut polytopes yield an infinite series of d-dimensional 0/1-polytopes with (roughly) $c^{\sqrt{d}}$ vertices (for some constant c) and graph-density one.

In this paper, we investigate the question for the graph-density of a typical (i.e., random) 0/1-polytope. The (perhaps surprising) result is that in fact the high density of the graphs of several 0/1-polytopes important in polyhedral combinatorics (such as the cut polytopes) is not atypical at all. Our main result is the following theorem, where $\operatorname{Exp}[\,\cdot\,]$ denotes the expected value.

Theorem 1. *Let $n : \mathbb{N} \longrightarrow \mathbb{N}$ be a function, and let $P_{d,n(d)} := \operatorname{conv} X_{d,n(d)}$ with an $n(d)$-element subset $X_{d,n(d)}$ of $\{0,1\}^d$ that is chosen uniformly at random. Denote by $\Delta_{d,n(d)}$ the density of the graph of $P_{d,n(d)}$.*

(i) If there is some $\varepsilon > 0$ such that $n(d) \le (\sqrt{2} - \varepsilon)^d$ for all sufficiently large d, then $\lim_{d\to\infty} \operatorname{Exp}[\,\Delta_{d,n(d)}\,] = 1$.
(ii) If there is some $\varepsilon > 0$ such that $n(d) \ge (\sqrt{2} + \varepsilon)^d$ for all sufficiently large d, then $\lim_{d\to\infty} \operatorname{Exp}[\,\Delta_{d,n(d)}\,] = 0$.

There is a similar threshold phenomenon for the volumes of random 0/1-polytopes. Let $\tilde{P}_{d,n(d)}$ be the convex hull of $n(d)$ points in $\{0,1\}^d$ that are chosen independently uniformly at random (possibly with repetitions). Dyer, Füredi, and McDiarmid [2] proved that the limit (for $d \to \infty$) of the expected value of the d-dimensional volume of $\tilde{P}_{d,n(d)}$ is zero if, for some $\varepsilon > 0$, $n(d) \le (\frac{2}{\sqrt{e}} - \varepsilon)^d$ holds for all sufficiently large d, and it is one if, for some $\varepsilon > 0$, $n(d) \ge (\frac{2}{\sqrt{e}} + \varepsilon)^d$ holds for all sufficiently large d. Due to $\frac{2}{\sqrt{e}} < 1.214$ and $\sqrt{2} > 1.414$, one can deduce (we omit the details) the following result from this and Theorem 1. It may be a bit surprising due to the fact that the only d-dimensional 0/1-polytope with d-dimensional volume equal to one is the 0/1-cube $\operatorname{conv}\{0,1\}^d$, which has only graph-density $\frac{d}{2^d-1}$.

Corollary 1. *For every $\delta > 0$ there are (infinitely many) 0/1-polytopes with both graph density and volume at least $(1 - \delta)$.*

Another threshold result that is related to our work is due to Füredi [4]. He showed that, in the setting of Theorem 1, the limit (for $d \to \infty$) of the probability that $P_{d,n(d)}$ contains the center of the 0/1-cube is zero if, for some $\varepsilon > 0$, $n(d) \le (2 - \varepsilon) \cdot d$ holds for all sufficiently large d, and it is one if, for some $\varepsilon > 0$, $n(d) \ge (2 + \varepsilon) \cdot d$ holds for all sufficiently large d. The material in Sections 2.2, 2.3, and 2.4 of our paper is very much inspired by Füredi's work.

The aim of Sections 2 and 3 is to prove Theorem 1. Since it is a bit more convenient, we switch from 0/1-polytopes to polytopes whose vertices have coordinates in $\{-1,+1\}$ (*±1-polytopes*). Recalling that the density of a graph equals the probability of a randomly chosen pair of its nodes to be adjacent, Propositions 4 and 5 (Section 3), together with Proposition 3, imply Theorem 1 (with the ε's in Propositions 4 and 5 replaced by $\log \frac{\sqrt{2}}{\sqrt{2}-\varepsilon}$ and $\log \frac{\sqrt{2}+\varepsilon}{\sqrt{2}}$, respectively).

We close with a few remarks in Section 4.

2 The Long-Edge Probability $\tau(k, m)$

We define $\mathrm{Q}_d := \{-1, +1\}^d$ and $\mathrm{Q}_d^\star := \mathrm{Q}_d \setminus \{-\mathbf{1}, \mathbf{1}\}$ (where $\mathbf{1}$ is the all-one vector). For $v, w \in \mathrm{Q}_d$, denote by $\mathrm{Q}(v, w)$ the subset of all points in Q_d that agree with v and w in all components, where v and w agree. Thus, $\mathrm{Q}(v, w)$ is the vertex set of the smallest face of $\operatorname{conv} \mathrm{Q}_d$ containing v and w. The dimension of this face is

$$\operatorname{dist}(v, w) \;:=\; \#\{\; i \in \{1, \ldots, d\} \;:\; v_i \neq w_i \;\}$$

(the *Hamming distance* of v and w). Let $\mathrm{Q}^\star(v, w) := \mathrm{Q}(v, w) \setminus \{v, w\}$.

We refer to [12] for all notions and results from polytope theory that we rely on. For a polytope P, we denote by $\mathrm{V}(P)$ and $\mathrm{E}(P)$ the sets of vertices and edges of P, respectively. Recall that, for $X \subseteq \mathrm{Q}_d$, we have $\mathrm{V}(\operatorname{conv} X) = X$.

The following fact is essential for our treatment. It can easily be deduced from elementary properties of convex polytopes.

Lemma 1. *For two vertices v and w of a ± 1-polytope $P \subset \mathbb{R}^d$ we have*

$$\{v, w\} \in \mathrm{E}(P) \iff \operatorname{conv}\{v, w\} \cap \operatorname{conv}(P \cap \mathrm{Q}^\star(v, w)) = \varnothing \;.$$

Throughout this section, let $Y_{k,m} \in \binom{\mathrm{Q}_k^\star}{m}$ (the m-element subsets of $\mathrm{Q}_k^\star$) be drawn uniformly at random and define

$$\tau(k, m) \;:=\; \operatorname{Prob}[\operatorname{conv}(Y_{k,m}) \cap \operatorname{conv}\{-\mathbf{1}, \mathbf{1}\} = \varnothing\,] \;.$$

Thus, $\tau(k, m)$ is the probability that the "long edge" $\operatorname{conv}\{-\mathbf{1}, \mathbf{1}\}$ is an edge of the polytope $\operatorname{conv}(Y_{k,m} \cup \{-\mathbf{1}, \mathbf{1}\})$. The next lemma follows from Lemma 1.

Lemma 2. *Let $X_{d,n} \in \binom{\mathrm{Q}_d}{n}$ be chosen uniformly at random, defining the polytope $P_{d,n} := \operatorname{conv} X_{d,n}$. Choose a two-element subset $\{v, w\}$ of $X_{d,n}$ uniformly at random. Then, for every $k \in \{1, \ldots, d\}$ and $m \in \{0, \ldots, \min\{2^k - 2, n - 2\}\}$, we have the equation*

$$\operatorname{Prob}[\,\{v, w\} \in \mathrm{E}(P_{d,n}) \mid \operatorname{dist}(v, w) = k, \#(X_{d,n} \cap \mathrm{Q}^\star(v, w)) = m\,] \;=\; \tau(k, m) \;.$$

Via Lemma 2, asymptotic bounds on $\tau(k, m)$ will turn out to be important for the proofs in Section 3. In fact, we will basically compute (or estimate) the probability $\pi(d, n)$ (see Section 3) that two randomly chosen vertices of a d-dimensional random ± 1-polytope with n vertices are adjacent by partitioning the probability space into the events "$\operatorname{dist}(v, w) = k$ and $\#(X_{d,n} \cap \mathrm{Q}^\star(v, w)) = m$" for all $k \in \{1, \ldots, d\}$ and $m \in \{0, \ldots, \min\{2^k - 2, n - 2\}\}$.

For the study of $\tau(k, m)$, it is convenient to consider the conditional probability

$$\alpha(k, m) \;:=\; \operatorname{Prob}[\operatorname{conv}(Y_{k,m}) \cap \operatorname{conv}\{-\mathbf{1}, \mathbf{1}\} = \varnothing \mid Y_{k,m} \cap (-Y_{k,m}) = \varnothing\,] \;,$$

which is related to $\tau(k, m)$ in the following way.

Lemma 3. *For $0 \leq m \leq 2^k - 2$ we have*

$$\tau(k,m) \;=\; \frac{\binom{2^{k-1}-1}{m} \cdot 2^m}{\binom{2^k-2}{m}} \cdot \alpha(k,m) \;.$$

Proof. Clearly, $\operatorname{conv}(Y_{k,m}) \cap \operatorname{conv}\{-\mathbf{1}, \mathbf{1}\} = \varnothing$ implies $Y_{k,m} \cap (-Y_{k,m}) = \varnothing$. Thus, the statement in the lemma is due to the fact that the number of sets $Y' \in \binom{\mathrm{Q}_k^\star}{m}$ with $Y' \cap (-Y') = \varnothing$ is $\binom{2^{k-1}-1}{m} \cdot 2^m$.

We will first show that $\alpha(k,m)$ can be interpreted as a conditional probability that a random m-element subset of a certain vector configuration in $\mathbb{R}^{k-1}$ does not contain the origin in its convex hull (Section 2.1). The latter probability is then related to the expected number of chambers in a certain random hyperplane arrangement. This number of chambers is finally estimated via a well-known bound due to Harding (Section 2.2).

As a point of reference for the proofs in Section 3, let us state the following monotonicity result here, whose (straightforward) proof we omit.

Lemma 4. *For $0 \leq m \leq 2^k - 3$, we have $\tau(k,m) \geq \tau(k,m+1)$.*

2.1 The Vector Configuration $\mathcal{V}_r$

Let $\varphi : \mathbb{R}^{r+1} \longrightarrow H_{\mathbf{1}} \longrightarrow \mathbb{R}^r$ denote the orthogonal projection of $\mathbb{R}^{r+1}$ onto the hyperplane $H_{\mathbf{1}} := \{x \in \mathbb{R}^{r+1} : \mathbf{1}^T x = 0\}$, followed by the orthogonal projection to the first r coordinates. We denote by $\mathcal{V}_r := \varphi(\mathrm{Q}_{r+1}^\star)$ the image of $\mathrm{Q}_{r+1}^\star$ under the projection φ. We omit the simple proof of the following result.

Lemma 5. *The projection φ is one-to-one on $\mathrm{Q}_{r+1}^\star$.*

Lemma 6. *For $Z_{r,m} \in \binom{\mathcal{V}_r}{m}$ chosen uniformly at random, we have*

$$\alpha(r+1,m) \;=\; \operatorname{Prob}[\,\mathbf{0} \notin \operatorname{conv}(Z_{r,m}) \mid Z_{r,m} \cap (-Z_{r,m}) = \varnothing\,] \;.$$

Proof. Since $\operatorname{conv} Y_{k,m} \cap \operatorname{conv}\{-\mathbf{1}, \mathbf{1}\} = \varnothing$ holds if and only if $\mathbf{0} \notin \operatorname{conv} \varphi(Y_{k,m})$ holds, the claim follows from Lemma 5 (because $Y_{k,m} \cap (-Y_{k,m}) = \varnothing$ is equivalent to $\varphi(Y_{k,m}) \cap (-\varphi(Y_{k,m})) = \varnothing$).

With $\mathcal{V}_r^+ := \varphi\{v \in \mathrm{Q}_{r+1}^\star : v_{r+1} = +1\}$, we have $\mathcal{V}_r = \mathcal{V}_r^+ \cup (-\mathcal{V}_r^+)$ and $\mathcal{V}_r^+ \cap (-\mathcal{V}_r^+) = \varnothing$. For any fixed finite subset $S \subset \mathbb{R}^r$, and a uniformly at random chosen $\varepsilon \in \{-1,+1\}^S$, denote $\alpha(S) := \operatorname{Prob}[\,\mathbf{0} \notin \operatorname{conv}\{\varepsilon_s s : s \in S\}\,]$.

Lemma 7. *Let $Z_{r,m}^+ \in \binom{\mathcal{V}_r^+}{m}$ be chosen uniformly at random. Then we have*

$$\alpha(r+1,m) \;=\; \operatorname{Exp}[\,\alpha(Z_{r,m}^+)\,] \;.$$

Proof. This follows from Lemma 6.

2.2 Hyperplane Arrangements

For $s \in \mathbb{R}^r \setminus \{\mathbf{0}\}$ let $H(s) := \{x \in \mathbb{R}^r : s^T x = 0\}$. The two connected components of $\mathbb{R}^r \setminus H(s)$ are denoted by $H^+(s)$ and $H^-(s)$, where $s \in H^+(s)$. For a finite subset $S \subset \mathbb{R}^r \setminus \{\mathbf{0}\}$ denote by $\mathcal{H}(S) := \{H(s) : s \in S\}$ the *hyperplane arrangement* defined by S. The connected components of $\overline{\mathcal{H}(S)} := \mathbb{R}^r \setminus \bigcup_{s \in S} H(s)$ are the *chambers* of $\mathcal{H}(S)$. We denote the number of chambers of $\mathcal{H}(S)$ by $\chi(S)$.

Observation 1 *Let C be a chamber of $\mathcal{H}(S)$ for some finite subset $S \subset \mathbb{R}^r \setminus \{\mathbf{0}\}$. For each $s \in S$, we have either $C \subseteq H^+(s)$ or $C \subseteq H^-(s)$. Defining $\varepsilon(C)_s := +1$ in the first, and $\varepsilon(C)_s := -1$ in the second case, we may assign a* sign vector *$\varepsilon(C) \in \{-1, +1\}^S$ to each chamber C of $\mathcal{H}(S)$. This assignment is injective.*

Lemma 8. *For each finite subset $S \subset \mathbb{R}^r \setminus \{\mathbf{0}\}$, the following equation holds:*

$$\#\{\ \varepsilon \in \{-1, +1\}^S \ :\ \mathbf{0} \notin \operatorname{conv}\{\ \varepsilon_s s \ :\ s \in S\ \}\ \} \ = \ \chi(S)$$

Proof. Let $S \subset \mathbb{R}^r \setminus \{\mathbf{0}\}$ be finite. By the Farkas-Lemma (linear programming duality), for each $\varepsilon \in \{-1, +1\}^S$, we have $\mathbf{0} \notin \operatorname{conv}\{\varepsilon_s s : s \in S\}$ if and only if there is some $h \in \mathbb{R}^r$ such that $h^T(\varepsilon_s s) > 0$ holds for all $s \in S$, which in turn is equivalent to

$$h^T s \begin{cases} > \ 0 & \text{if } \varepsilon_s = +1 \\ < \ 0 & \text{if } \varepsilon_s = -1 \end{cases}$$

for all $s \in S$. Since the latter condition is equivalent to ε being the sign vector of some chamber of $\mathcal{H}(S)$, the statement of the lemma follows.

Lemma 7 and Lemma 8 immediately yield the following result.

Lemma 9. *For $Z^+_{r,m} \in \binom{\mathcal{V}^+_r}{m}$ chosen uniformly at random, we have*

$$\alpha(r+1, m) \ = \ \frac{1}{2^m} \cdot \operatorname{Exp}[\,\chi(Z^+_{r,m})\,] \ .$$

The following upper bound on $\chi(\cdot)$ will (via Lemma 9) yield upper bounds on $\alpha(\cdot, \cdot)$ that are sufficient for our needs. We denote $b(p, q) := \sum_{i=0}^{p} \binom{q}{i}$.

Theorem 2 (Harding, see Winder [11, p. 816]). *For $S \in \binom{\mathbb{R}^r \setminus \{\mathbf{0}\}}{m}$, we have*

$$\chi(S) \ \leq \ 2b(r-1, m-1) \ .$$

2.3 Bounds on $\tau(k, m)$

Proposition 1. *For $0 \leq m \leq 2^k - 2$ the following inequality holds:*

$$\tau(k, m) \ \leq \ \frac{b(k-2, m-1)}{2^{m-1}}$$

Proof. With $r = k-1$, Lemma 3, Lemma 9, and Theorem 2 yield this.

In fact, one can prove that, if m is not too large relative to k, then the bound of Proposition 1 is asymptotically sharp as k tends to infinity. Since we do not need the result here, we omit the proof which (next to the theorem of Winder's cited in Theorem 2) relies on the fact that the probability of an $l \times l$ matrix with entries from $\{-1, +1\}$ (chosen uniformly at random) being singular converges to zero for l tending to infinity (see [6]).

Proposition 2. *For $m(k) \in o\big(2^{\frac{k}{2}}\big)$, we have*

$$\lim_{k\to\infty} \left(\tau(k, m(k)) - \frac{b(k-2, m(k)-1)}{2^{m(k)-1}} \right) \;=\; 0 \;.$$

2.4 A Threshold for $\tau(k, m)$

For $x \in \mathbb{R}$, let

$$\Phi(x) \;:=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, \mathrm{d}t \;,$$

i.e., Φ is the density function of the normal distribution.

Lemma 10 (de Moivre-Laplace theorem). *For each $\mu \in \mathbb{R}$, the following holds:*

$$\lim_{q\to\infty} \frac{b\left(\lfloor \frac{q}{2} + \mu\sqrt{q} \rfloor, q\right)}{2^q} \;=\; \Phi(2\mu)$$

Theorem 3. *For each $\varepsilon > 0$, we have*

$$\lim_{k\to\infty} \tau\big(k, \lceil (2+\varepsilon)k \rceil\big) \;=\; 0 \;.$$

Proof. Let $\varepsilon > 0$ be fixed, and define, for each k, $m_\varepsilon^+(k) := \lceil (2+\varepsilon)k \rceil$.

Let $\delta > 0$ be arbitrarily small, and choose $\mu < 0$ such that

$$\Phi(2\mu) \;<\; \frac{\delta}{2} \;. \tag{1}$$

Due to $\lim\limits_{k\to\infty} \dfrac{m_\varepsilon^+(k)}{k} = 2 + \varepsilon$, we have, for large enough k,

$$k - 2 \;\leq\; \frac{m_\varepsilon^+(k) - 1}{2} + \mu\sqrt{m_\varepsilon^+(k) - 1} \;. \tag{2}$$

Due to Proposition 1, we have

$$\tau(k, m_\varepsilon^+(k)) \;\leq\; \frac{b(k-2, m_\varepsilon^+(k) - 1)}{2^{m_\varepsilon^+(k)-1}} \;. \tag{3}$$

Since $b(\cdot,\cdot)$ is monotonically increasing in the first component, (2) yields that the right-hand side of (3) is bounded from above by

$$\frac{b\Big(\frac{m_\varepsilon^+(k)-1}{2} + \mu\sqrt{m_\varepsilon^+(k) - 1}\;,\; m_\varepsilon^+(k) - 1\Big)}{2^{m_\varepsilon^+(k)-1}} \;. \tag{4}$$

By Lemma 10 (with q substituted by $m_\varepsilon^+(k)-1$), (4) may be bounded from above by $\Phi(2\mu)+\frac{\delta}{2}$ for all large enough k (because of $\lim_{k\to\infty} m_\varepsilon^+(k)=\infty$). Thus, from (1) we obtain

$$\tau(k, m_\varepsilon^+(k)) \; < \; \delta$$

for all large enough k.

Exploiting Proposition 2, one can also prove the following result. It complements Theorem 3, but since we will not need it in our treatment, we do not give a proof here.

Theorem 4. *For each $\varepsilon > 0$ we have*

$$\lim_{k\to\infty} \tau\big(k, \lfloor (2-\varepsilon)k \rfloor\big) \; = \; 1$$

3 The Edge Probability $\pi(d, n)$

Throughout this section, let the set $X_{d,n} \in \binom{\mathrm{Q}_d}{n}$ be drawn uniformly at random, $P_{d,n} := \operatorname{conv} X_{d,n}$, and let $\{v, w\} \in \binom{X_{d,n}}{2}$ be chosen uniformly at random as well. Our aim is to determine the probability

$$\pi(d, n) \; := \; \operatorname{Prob}[\, \{v, w\} \in \mathrm{E}(P_{d,n}) \,] \; .$$

Let us further denote

$$\pi_k(d, n) \; := \; \operatorname{Prob}[\, \{v, w\} \in \mathrm{E}(P_{d,n}) \mid \operatorname{dist}(v, w) = k \,] \; .$$

Since $\{v, w\}$ is uniformly distributed over $\binom{\mathrm{Q}_d}{2}$, the distance $\operatorname{dist}(v, w)$ has the same distribution as the number of positive components of a point chosen uniformly at random from $\mathrm{Q}_d \setminus \{-\mathbf{1}\}$. Therefore, the following equation holds.

Lemma 11.

$$\pi(d, n) \; = \; \frac{1}{2^d - 1} \sum_{k=1}^{d} \binom{d}{k} \pi_k(d, n)$$

The following result, stating that $\pi(d, \cdot)$ is monotonically increasing, is quite plausible. Its straightforward proof is omitted here.

Proposition 3. *The function $\pi(d, \cdot)$ is monotonically decreasing, i.e., for $3 \le n \le 2^d - 1$, we have $\pi(d, n) > \pi(d, n+1)$.*

The next result implies part (i) of Theorem 1 (see the remarks at the end of Section 1).

Proposition 4. *For each $\varepsilon > 0$, we have*

$$\lim_{d\to\infty} \pi\left(d, \left\lfloor 2^{\left(\frac{1}{2}-\varepsilon\right)d} \right\rfloor\right) \; = \; 1 \; .$$

Proof. Let $\varepsilon > 0$, and define $n_\varepsilon^-(d) := \lfloor 2^{\left(\frac{1}{2}-\varepsilon\right)d} \rfloor$. For each $\mu > 0$, denote

$$K_\mu^{\leq}(d) \;:=\; \{k \in \mathbb{Z} : 1 \leq k \leq \frac{d}{2} + \mu\sqrt{d}\}$$

and

$$\pi_\mu^-(d) \;:=\; \min\left\{\; \pi_k(d, n_\varepsilon^-(d)) \;:\; k \in K_\mu^{\leq}(d) \;\right\} .$$

Then, due to Lemma 11, we have

$$\pi(d, n_\varepsilon^-(d)) \;\geq\; \sum_{k \in K_\mu^{\leq}(d)} \frac{\binom{d}{k}}{2^d} \cdot \pi_\mu^-(d) \;.$$

For every $\nu > 0$, this implies (by Lemma 10) that

$$\pi(d, n_\varepsilon^-(d)) \;\geq\; (\Phi(2\mu) - \nu) \cdot \pi_\mu^-(d) \tag{5}$$

holds for all large enough d. Therefore, it remains to prove, for all $\mu > 0$,

$$\lim_{d\to\infty} \pi_\mu^-(d) \;=\; 1 \;. \tag{6}$$

With

$$\xi_k \;:=\; \operatorname{Prob}[\, X_{d,n_\varepsilon^-(d)} \cap \mathrm{Q}^\star(v,w) = \varnothing \mid \operatorname{dist}(v,w) = k \,] \;,$$

we have, for each $k \in K_\mu^{\leq}(d)$,

$$\pi_k(d, n_\varepsilon^-(d)) \;\geq\; \xi_k \;\geq\; \xi_{\lfloor \frac{d}{2} + \mu\sqrt{d} \rfloor} \tag{7}$$

(see Lemma 1). Clearly,

$$\operatorname{Exp}[\, \#(X_{d,n_\varepsilon^-(d)} \cap \mathrm{Q}^\star(v,w)) \mid \operatorname{dist}(v,w) = k \,] \;=\; \frac{2^k - 2}{2^d - 2} \cdot (n_\varepsilon^-(d) - 2) \;,$$

and thus, the estimation

$$\operatorname{Exp}[\, \#(X_{d,n_\varepsilon^-(d)} \cap \mathrm{Q}^\star(v,w)) \mid \operatorname{dist}(v,w) = k \,] \;\leq\; 2^{k-(\frac{1}{2}+\varepsilon)d} \;,$$

hold for each k. By Markov's inequality, this implies

$$\operatorname{Prob}[\, \#(X_{d,n_\varepsilon^-(d)} \cap \mathrm{Q}^\star(v,w)) \geq d \cdot 2^{k-(\frac{1}{2}+\varepsilon)d} \mid \operatorname{dist}(v,w) = k \,] \;\leq\; \frac{1}{d} \tag{8}$$

for each d and k. For $k = \lfloor \frac{d}{2} + \mu\sqrt{d} \rfloor$, (8) yields

$$\operatorname{Prob}[\, \#(X_{d,n_\varepsilon^-(d)} \cap \mathrm{Q}^\star(v,w)) \geq d \cdot 2^{\mu\sqrt{d}-\varepsilon d} \mid \operatorname{dist}(v,w) = \lfloor \frac{d}{2} + \mu\sqrt{d} \rfloor \,] \\ \leq\; \frac{1}{d} \tag{9}$$

for all d. Since $d \cdot 2^{\mu\sqrt{d}-\varepsilon d} < 1$ holds for large enough d, (9) implies $\xi_{\lfloor \frac{d}{2}+\mu\sqrt{d} \rfloor} \geq 1 - \frac{1}{d}$ for large enough d. Therefore,

$$\lim_{d\to\infty} \xi_{\lfloor \frac{d}{2}+\mu\sqrt{d} \rfloor} \;=\; 1$$

holds, which, by (7), finally implies (6).

The next result yields part (ii) of Theorem 1 (see the remarks at the end of Section 1).

Proposition 5. *For each $\varepsilon > 0$, we have*

$$\lim_{d\to\infty} \pi\left(d, \left\lceil 2^{\left(\frac{1}{2}+\varepsilon\right)d}\right\rceil\right) \;=\; 0\ .$$

Proof. Let $\varepsilon > 0$, and define $n_\varepsilon^+(d) := \left\lceil 2^{\left(\frac{1}{2}+\varepsilon\right)d}\right\rceil$. For each $\mu > 0$, denote

$$K_\mu^{\geq}(d) \;:=\; \{k \in \mathbb{Z} : \frac{d}{2} - \mu\sqrt{d} \leq k \leq d\}\ ,$$

and define

$$\pi_\mu^+(d) \;:=\; \max\{\ \pi_k(d, n_\varepsilon^+(d)) \;:\; k \in K_\mu^{\geq}(d)\ \}\ . \tag{10}$$

Then, due to Lemma 11, we have

$$\pi(d, n_\varepsilon^+(d)) \;\leq\; 2 \cdot \sum_{k=1}^{\lfloor \frac{d}{2} - \mu\sqrt{d}\rfloor} \frac{\binom{d}{k}}{2^d} \;+\; \pi_\mu^+(d)\ .$$

Thus, for every $\nu > 0$, by Lemma 10,

$$\pi(d, n_\varepsilon^+(d)) \;\leq\; \Phi(-2\mu) + \nu + \pi_\mu^+(d)$$

holds for all large enough d. Therefore, it remains to prove, for all $\mu > 0$,

$$\lim_{d\to\infty} \pi_\mu^+(d) \;=\; 0\ . \tag{11}$$

For $k \in \{1, \ldots, d\}$ and $m \in \{0, \ldots, 2^k - 2\}$, we define

$$\xi_k(m) \;:=\; \mathrm{Prob}[\,\#(X_{d,n_\varepsilon^+(d)} \cap \mathrm{Q}^\star(v,w)) = m \mid \mathrm{dist}(v,w) = k\,]$$

(i.e., $\xi_k(0) = \xi_k$ in the proof of Proposition 4). Then we have (see Lemma 2)

$$\pi_k(d, n_\varepsilon^+(d)) \;=\; \sum_{m=0}^{2^k-2} \xi_k(m)\tau(k,m)\ . \tag{12}$$

Since $\tau(k,\cdot)$ is monotonically non-increasing by Lemma 4, we thus can estimate

$$\pi_k(d, n_\varepsilon^+(d)) \;\leq\; \sum_{m=0}^{3k-1} \xi_k(m) \;+\; \tau(k, 3k)\ ,$$

for each $k \in K_\mu^{\geq}(d)$. This yields, again for for each $k \in K_\mu^{\geq}(d)$,

$$\begin{aligned}\pi_k(d, n_\varepsilon^+(d)) \;\leq\; & 3d \cdot \max\{\ \xi_k(m) \;:\; 0 \leq m \leq 3d-1\ \} \\ & + \max\{\ \tau(k', 3k') \;:\; k' \in K_\mu^{\geq}(d)\ \}\ .\end{aligned} \tag{13}$$

According to Theorem 3,

$$\lim_{d\to\infty} \max\{\tau(k', 3k') : k' \in K_\mu^{\geq}(d)\} \;=\; 0$$

holds. Hence, by (13) and (10), equation (11) can be proved by showing

$$\lim_{d\to\infty} \left(3d \cdot \max\{\; \xi_k(m) \;:\; 0 \leq m \leq 3d-1,\; k \in K_\mu^{\geq}(d)\;\}\right) \;=\; 0\,. \tag{14}$$

Let us first calculate (using the notation $(a)_b := a(a-1)\cdots(a-b+1)$)

$$\begin{aligned}\xi_k(m) \;&=\; \frac{\binom{2^k-2}{m}\binom{2^d-2^k}{n_\varepsilon^+(d)-m-2}}{\binom{2^d-2}{n_\varepsilon^+(d)-2}} \\ &=\; \binom{2^k-2}{m} \cdot \frac{(2^d-2^k)_{n_\varepsilon^+(d)-m-2}}{(2^d-2)_{n_\varepsilon^+(d)-2}} \cdot \frac{(n_\varepsilon^+(d)-2)!}{(n_\varepsilon^+(d)-m-2)!}\,,\end{aligned} \tag{15}$$

where the left, the middle, and the right factor of (15) may be bounded from above by $(2^d)^m$, $(2^d)^2 \cdot \left(\frac{2^d-2^k}{2^d}\right)^{n_\varepsilon^+(d)}$, and $(2^d)^m$, respectively. Thus, we obtain, for $0 \leq m \leq 3d-1$,

$$\xi_k(m) \;\leq\; 2^{\mathrm{const}\cdot d^2} \cdot \left(1 - \frac{1}{2^{d-k}}\right)^{n_\varepsilon^+(d)}\,. \tag{16}$$

For $k \in K_\mu^{\geq}(d)$, we have

$$\begin{aligned}\left(1 - \frac{1}{2^{d-k}}\right)^{n_\varepsilon^+(d)} \;&\leq\; \left(1 - \frac{1}{2^{\frac{d}{2}+\mu\sqrt{d}}}\right)^{2^{\left(\frac{1}{2}+\varepsilon\right)d}} \\ &=\; \left[\left(1 - \frac{1}{2^{\frac{d}{2}+\mu\sqrt{d}}}\right)^{2^{\frac{d}{2}+\mu\sqrt{d}}}\right]^{2^{\varepsilon d-\mu\sqrt{d}}}\,.\end{aligned} \tag{17}$$

For d tending to infinity, the expression in the square brackets of (17) converges to $\frac{1}{e} < \frac{1}{2}$ (where $e = 2.7182\cdots$ is Euler's constant). Therefore, (17) and (16) imply $\xi_k(m) \leq 2^{\mathrm{const}\cdot d^2} \cdot (1/2)^{2^{\varepsilon d-\mu\sqrt{d}}}$ (for $k \in K_\mu^{\geq}(d)$, $0 \leq m \leq 3d-1$, and for large enough d). This finally yields (14), and therefore completes the proof.

4 Remarks

The threshold for the function $\tau(\cdot,\cdot)$ described in Theorems 3 and 4 is much sharper than we needed for our purposes (proof of Proposition 5). The sharper result may, however, be useful in investigations of more structural properties of the graphs of random 0/1-polytopes. A particularly interesting such question is whether these graphs have good expansion properties with high probability.

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